

On internal wave–shear flow resonance in shallow water

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The work is concerned with long nonlinear internal waves interacting with a shear flow localized near the sea surface. The study is focused on the most intense *resonant* interaction occurring when the phase velocity of internal waves matches the flow velocity at the surface. The perturbations of the shear flow are considered as ‘vorticity waves’, which enables us to treat the wave–flow resonance as the resonant wave–wave interaction between an internal gravity mode and the vorticity mode. Within the weakly nonlinear long-wave approximation a system of evolution equations governing the nonlinear dynamics of the waves in resonance is derived and an asymptotic solution to the basic equations is constructed. At resonance the nonlinearity of the internal wave dynamics is due to the interaction with the vorticity mode, while the wave’s own nonlinearity proves to be negligible. The equations derived are found to possess solitary wave solutions of different polarities propagating slightly faster or slower than the surface velocity of the shear flow. The amplitudes of the ‘fast’ solitary waves are limited from above; the crest of the limiting wave forms a sharp corner. The solitary waves of amplitude smaller than a certain threshold are shown to be stable; ‘subcritical’ localized pulses tend to such solutions. The localized pulses of amplitude exceeding this threshold form infinite slopes in finite time, which indicates wave breaking.

1. Introduction

Continuing progress in the remote sensing of the ocean surface makes the study of links between the processes in the water interior and their surface signatures one of the ‘hottest’ topics of physical oceanography today. Internal gravity waves, being widespread in all natural basins, remain the only ‘internal’ process having numerous and well-documented surface manifestations supported by *in situ* measurements (e.g. Apel *et al.* 1985). The interest in the physical mechanisms of such manifestations has resulted in a vast literature (see e.g. Robinson 1985 for a review of the basic mechanisms). Recently a new mechanism for the amplification of internal wave manifestations due to their resonance with a thin subsurface shear current was

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considered in Voronovich & Shrira (1996) within the framework of a linear theory. The present work develops a nonlinear description of such resonance and, in particular, shows its prime importance not only in the context of surface signatures but also for internal wave dynamics in shallow water.

Internal gravity waves in themselves have been a subject of intense studies since the beginning of the century (see e.g. Leblond & Mysak 1979; Phillips 1977; Turner 1973), which, now, after reaching a peak in the seventies, is experiencing a kind of renaissance, especially in the context of coastal waters. The growing number of field observations in the coastal zone, their improved quality and, especially, new capabilities of computer data processing has created a demand for new theoretical models that better explain the variety of features of internal wave dynamics. By now, the basic features of internal wave evolution in the coastal waters have been well established experimentally: the waves are long compared to the typical water depth; the in-shore propagating waves are, as a rule, much more pronounced; they exhibit essentially nonlinear behaviour; solitary-wave-type patterns are dominant quite often in the wave dynamics, although bore-like structures might occur as well (Ostrovsky & Stepanyants 1989; Serebryany 1993). The basic theoretical model used in the last two decades to describe the field evolution is the Korteweg–de Vries equation with variable coefficients, due to the bottom topography and large-scale inhomogeneity coefficients (see e.g. Grimshaw 1986). Modifications allowing for Earth's rotation were developed to describe the nonlinear evolution of lower-frequency waves or waves at larger times (Ostrovsky 1978). In the specific situation with vanishing quadratic nonlinearity taking account of a cubic one leads to the cubic or mixed Korteweg–de Vries (KdV) equations. The role of shear currents within the framework of such models although important is still merely quantitative: the coefficients of the underlying equation depend on the vertical profile of the currents (Maslowe & Redekopp 1980; Grimshaw 1997).

To ensure stability of the flows under consideration it is usually presumed that the Richardson number $Ri = [N(z)/U'(z)]^2$, where $N(z)$, $U'(z)$ are the Brunt-Väisälä frequency and the mean current vertical gradient, exceeds $1/4$ everywhere. The opposite inequality, i.e. the Richardson number being smaller than $1/4$, means that owing to the mean shear current the forces due to the inhomogeneous vorticity field dominate those of buoyancy which are due to the density stratification. However, even in the limit of small Richardson number, when the shear effects prevail, the current does not necessarily become unstable, although the nature of the wave dynamics changes drastically. The main new feature, which is typical of internal waves at small Richardson numbers, is the critical layer which means that there is a depth where the wave phase velocity matches that of the current. The vast literature devoted to the critical layer problem is confined to the situation of monochromatic or, at best, nearly monochromatic waves in a deep fluid (see e.g. Leblond & Mysak 1979; Craik 1985 and references therein). In terms of the linear spectral problem for guided internal waves (the Taylor–Goldstein equation with the corresponding boundary conditions at the surface and the bottom) the presence of critical layers leads to appearance of singular modes of continuous spectrum. The problem of treating these singularities is commonly circumvented by taking into account either viscous or nonlinear effects prevailing in the *thin* critical layer. In our context the problem lies in the fact that the shallow water waves usually are far from being monochromatic and the description of the critical layer interactions for such waves encounters insurmountable difficulties in the generic case, especially for non-planar motions.

However, in the geophysically most relevant situations, those with the shear due to wind drift current localized near the surface, solutions to the basic equations may be

sought in terms of an asymptotic expansion in powers of the natural small parameter ϵ characterizing the smallness of the current depth compared to a typical wavelength. The remarkable fact that allows us to progress is that arbitrary perturbations of such a thin shear current comprising an aggregate of the singular modes behave at certain timescales as if they were a *single discrete mode* having no singularities at the leading order in ϵ (Shrira 1989; Voronovich, Shrira & Stepanyants 1997). Next-order solutions usually exhibit a logarithmic singularity for strictly plane motions and a pole-like singularity for non-planar ones. The presence of stratified fluid below very weakly affects the properties of this single mode: a localized current with a strong shear can support wave-like motions even in the absence of any stratification. We shall call such modes *vorticity waves* as their existence is due to the mean flow inhomogeneous vorticity field. The vorticity waves are weakly dispersive and their phase speed tends to the flow velocity at the surface in the long-wave limit. Treating these perturbations as waves not only greatly simplifies their description, but it also makes it possible to describe *wave–flow* interaction in terms of *wave–wave* interaction.

Thus, in typical oceanic conditions a stratified shear flow characterized by a thin wind-driven subsurface current can support wave motions of two physically different types, vorticity and internal gravity waves which interact weakly or strongly, depending on the particular environment features. If the phase speed of a certain internal wave mode coincides with that of the vorticity mode, i.e. approximately equals the surface velocity of the flow, a resonance of internal waves with shear flow occurs and a particularly strong interaction, which we are interested in, takes place. In terms of wave–wave interactions this resonance is of the *direct resonance* type (Akylas & Benney 1980, 1982) which means that in resonant conditions both the celerities and vertical structure of the modes are close to each other. *Direct resonance* is qualitatively different from that between two internal wave modes originating from two different thermoclines, studied by Gear & Grimshaw (1984) and Gear (1985), because in the latter case only the phase speeds of the modes match while the vertical structure remains different and a traditional KdV-type theory can be applied even at the resonant conditions. As shown below this is not the case for the direct resonance and a new theory is required which is the subject of this paper.

A linear theory of an internal wave–shear flow resonant interaction was first developed by Reutov (1990) and later by Voronovich & Shrira (1996). The latter work established the resonance to be of the *direct resonance* type, i.e. the vertical structures of the interacting modes are close to each other, and found that the process leads to a significant amplification of the wave motion near the surface. However, the question of whether there exists a limitation on the motion amplitudes as well as a number of other physically important questions could not be resolved within the framework of the linear theory. In the present work we take the natural next step: using a standard technique for the wave–wave interactions we derive a simple nonlinear model describing the phenomenon of the resonance. The main implication of the latter for internal wave dynamics is that it makes the dynamics of even relatively weak internal waves essentially nonlinear.

We start in §2 with the derivation of the set of equations governing nonlinear dynamics of quasi-plane internal waves in stratified shear flow assuming the shear to be localized near the surface and the waves to be long compared to the fluid depth. In the absence of a resonance with the vorticity mode the equations are shown to reduce to the famous Kadomtsev–Petviashvili (KP) equation. The standard procedure breaks down in the presence of the resonance, because of the strong coupling of an internal mode and the vorticity modes. In §3 we study their nonlinear resonant interaction by

means of an asymptotic analysis and arrive at the set of two coupled equations for normalized wave amplitudes a and b of the internal and vorticity modes, respectively,

$$\left. \begin{aligned} (a_t + \Delta a_x + a_{xxx} - b_x)_x + a_{yy} &= 0, \\ b_t + 2bb_x - a_x &= 0, \end{aligned} \right\} \quad (1.1)$$

where Δ is a free parameter corresponding to a mismatch in the phase speeds of the coupled waves. In §4 we look for the basic solutions to (1.1) in the form of steady solitary waves and find plane solitary wave solutions of two different types. Solutions of the first type propagate with velocities greater than that of the current and are characterized by different polarities of a and b . Their amplitudes are limited by a critical value at which the solitary wave exhibits a sharp corner at the crest. Solitary waves of the second type have velocities smaller than the flow speed at the surface and the polarities of a and b are identical.

In §5 we study stability of plane solitary waves with respect to small plane perturbations and find the conditions for the solitary waves of the first type to become unstable. The development of this nonlinear instability, as well as the generic evolution of localized perturbations with ‘supercritical’ amplitudes, leads to the formation of vertical slopes and, thus, to wave breaking in finite time. The problem of the existence of solitons with critical amplitudes and the aforementioned scenarios of nonlinear soliton dynamics seem to be very similar to those studied within the framework of Whitham’s integro–differential equation modelling wave breaking by Fornberg & Whitham (1978) or of the integrable equation for the so-called *peaked* solitons by Camassa & Holm (1993). We should mention that the existence of solitons with critical and supercritical amplitudes was also investigated using the primitive equations for some particular stratified shear flows (Amick & Turner 1986; Pullin & Grimshaw 1988).

The results obtained are briefly discussed in the concluding §6. In the Appendix we study some features of oblique solitary wave solutions to (1.1), and we show that no solitary waves localized in *all* directions can exist within the framework of the model (1.1) because of the inextinguishable resonance with the small-amplitude linear waves, in accordance with the earlier results by Voronovich *et al.* (1997).

2. Formulation of the problem

We study the particular case of long-wave dynamics in inviscid incompressible stratified fluid of a finite constant depth in the presence of a comparatively strong shear flow. Fluid density stratification is supposed to be smooth and the shear flow to be localized in the thin subsurface layer of typical width h and to have no inflection points in the velocity profile $U(z)$ † (see figure 1). The total fluid depth H is assumed to be much greater than h but small compared to the typical wavelength.

In the Cartesian frame with the x -axis directed streamwise and the z -axis vertically upward the non-dimensional governing equations (the Euler, the mass conservation and the continuity ones) in the standard Boussinesq approximation are (e.g. Leblond

† The latter condition is to ensure stability of the flows under consideration with respect to *all* perturbations, not necessarily long-wave ones. The linear stability of such flows may be seen from the dispersion relation derived in Voronovich & Shrira (1996) for the case of the simplest piecewise linear model or from ‘energy sign’ considerations in the spirit of Craik (1985, §2.3).

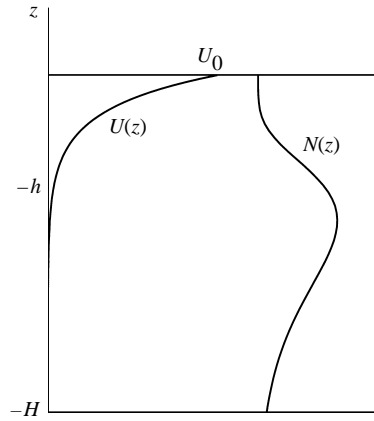


FIGURE 1. Flow geometry and notation.

& Mysak 1979; Phillips 1977)

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla \frac{p}{\rho_0} + \frac{\rho}{\rho_0} \mathbf{z} = 0, \quad (2.1a)$$

$$\rho_t + wN^2 + (\mathbf{u} \cdot \nabla) \rho = 0, \quad (2.1b)$$

$$\nabla \cdot \mathbf{u} = 0. \quad (2.1c)$$

Here p , and ρ are the pressure and density perturbations, $\rho_0(z)$ is the equilibrium density distribution, \mathbf{z} is a unit vertical vector, $\nabla = \{\partial_x, \partial_y, \partial_z\}$ is the gradient operator, subscripts denote the corresponding derivatives and the fluid velocity \mathbf{u} contains both the mean flow and perturbations

$$\mathbf{u}(x, y, z, t) = \{U(z) + u, v, w\}.$$

Use of the standard Boussinesq approximation is the reason for the neglect of nonlinear terms including density perturbations as well as the mean density variations with the depth (e.g. Leblond & Mysak 1979; Phillips 1977).

The equations were made non-dimensional by employing the scaling

$$\left. \begin{aligned} \{x', y', z'\} &= H_0 \{x, y, z\}, & t' &= \frac{H_0}{U_0} t, & p' &= \rho_* U_0^2 p, \\ \{\mathbf{u}', U'\} &= U_0 \{\mathbf{u}, U\}, & \rho' &= \rho_* \frac{U_0^2}{gH_0} \rho, & \rho'_0 &= \rho_* \rho_0; \\ N' &= \frac{U_0}{H_0} N, \end{aligned} \right\} \quad (2.2)$$

where H_0 is the typical fluid depth[†], U_0 is a typical, say maximal, value of the flow speed, ρ_* is a typical constant fluid density, say that in the upper mixed layer, N is the Brunt–Väisälä frequency

$$N^2(z) = -g \frac{\rho_{0z}}{\rho_0}$$

and primes denote dimensional variables.

[†] The non-dimensional depth H can be set equal to unity without loss of generality; however we prefer to preserve H , presuming $H \approx 1$, to facilitate generalization of the results for the case of varying depth to be considered elsewhere.

The standard boundary conditions

$$w = 0 \quad \text{at} \quad z = -H, \quad z = 0 \quad (2.3)$$

should be applied to the solutions of (2.1), the latter being the ‘no-flux’ condition at the rigid boundary and the former, the so-called ‘rigid-lid’ condition, naturally arising in studies of internal waves owing to the negligible smallness of the wave Froude numbers (Leblond & Mysak 1979; Phillips 1977).

We are interested in the dynamics of long quasi-plane waves in a stratified shear flow, i.e. those with the spanwise scale much larger than the streamwise one which, in turn, is much larger than the characteristic stratification scale and the fluid depth H . It is well-known that long waves in a smoothly stratified fluid with a shear flow are described by an infinite set of vertical eigenmodes specified by the linear boundary-value problem, the amplitudes of which satisfy the KdV equations for a plane (x, z) motion and the KP equations for a weakly three-dimensional (x, y, z) motion. To derive such equations, one should apply the multiple-scale method by introducing ‘slow’ space–time variables

$$X = \mu(x - ct), \quad Y = \mu^2 y, \quad T = \mu^3 t, \quad (2.4)$$

where $\mu \ll 1$ and c is the limiting long-wave speed to be determined, and looking for solutions to (2.1)–(2.3) in the form of an asymptotic expansion in powers of μ

$$\left. \begin{aligned} u &= \mu^2 A [(c - U)\Phi]_z + O(\mu^4), & p &= \mu^2 A \rho_0 (c - U)^2 \Phi_z + O(\mu^4), \\ v &= \mu^3 \left(\int A_Y dx \right) (c - U)\Phi_z + O(\mu^5), & \rho &= -\mu^2 A \rho_{0z} \Phi + O(\mu^4), \\ w &= -\mu^3 A_X (c - U)\Phi + O(\mu^5), \end{aligned} \right\} \quad (2.5)$$

where $A = A(X, Y, T)$ and $\Phi = \Phi(z)$. Such a representation singles out one of the modal functions $\Phi(z)$ satisfying the boundary-value problem with c as an eigenvalue,

$$\left. \begin{aligned} [(c - U)^2 \Phi_z]_z + N^2 \Phi &= 0, \\ \Phi(0) = \Phi(-H) &= 0, \end{aligned} \right\} \quad (2.6)$$

while the evolution of the mode amplitude $A(X, Y, T)$ is governed by the KP equation,

$$(\alpha A_T + \beta A_{XXX} + \gamma A A_X)_X + \delta A_{YY} = 0, \quad (2.7)$$

with the coefficients given by

$$\alpha = 2 \int_{-H}^0 (c - U) \Phi_z^2 dz, \quad \beta = \int_{-H}^0 (c - U)^2 \Phi^2 dz, \quad (2.8a,b)$$

$$\gamma = 3 \int_{-H}^0 (c - U)^2 \Phi_z^3 dz, \quad \delta = \int_{-H}^0 (c - U)^2 \Phi_z^2 dz. \quad (2.8c,d)$$

The boundary-value problem (2.6) for the smooth functions $U(z)$ and $N^2(z)$ possesses an infinite countable set of discrete eigenvalues $c = c_n$ ($n = 1, 2, \dots$), which correspond to the set of discrete linear modes $\Phi = \Phi_n$. The evolution equation (2.7) for a particular chosen n th mode is valid only when all other modes are far from resonance with the chosen one, i.e. the wave phase speed c_n does not coincide with that of any other mode c_m . This being the case, the modes are weakly coupled and do not interact at the leading orders. The nonlinear dynamics of such an isolated finite-amplitude internal

wave is then described completely by (2.7) or, for a plane wave ($A = A(X, T)$), by the KdV equation. The latter exhibits rather simple wave field dynamics which usually reduces to the generation of a number of solitary waves from an initial localized pulse and their successive interaction (e.g. Karpman 1975; Ablowitz & Clarkson 1991).

The resonance of modes originating from two different thermoclines in still fluid was considered by Gear & Grimshaw (1984) and Gear (1985), where a system of coupled KdV equations was derived for two plane ($A_{YY} = 0$) waves instead of (2.7). The analysis was based on the fact that near resonance the phase velocities of the waves are close to each other, $c_n \rightarrow c_m$ and $n \neq m$, but the mode functions Φ_n and Φ_m have different spatial structures. In contrast, in the presence of a shear flow the resonance $c_n \rightarrow c_m$ may occur, with the mode functions Φ_n and Φ_m coinciding identically at the leading order. This is due to the fact that a shear flow can support an additional wave mode, the vorticity wave.

To cast some light on the nature of the vorticity waves consider a simplified model consisting of a uniform fluid of total depth H and a shear flow with a piecewise-linear profile

$$U = \begin{cases} 1 + z/h & \text{at } -h \leq z \leq 0 \\ 0 & \text{at } -H \leq z \leq -h. \end{cases} \quad (2.9)$$

On substituting (2.9) into (2.6) and assuming $N^2 = 0$ the solution for the mode function is easily obtained as

$$\Phi = \Phi_v = \begin{cases} z & \text{at } -h < z \leq 0 \\ h \frac{z + H}{h - H} & \text{at } -H \leq z \leq -h, \end{cases} \quad (2.10a)$$

$$c = c_v = 1 - h/H. \quad (2.10b)$$

The solution found represents, in fact, a long wave having the maximum of its modal function at the vorticity jump at $z = -h$. These motions are often called *vorticity waves*. In (2.10b) and below subscript v stands for vorticity waves.

In the case of the smooth shear profile, the situation, though being much more mathematically complicated, still preserves some basic properties of the simplest model. The discrete modes are replaced by a continuous spectrum, which, for intermediate times and arbitrary long-wave perturbations still behaves like the discrete mode (2.10). The waves of the continuous spectrum form an *intermediate asymptotic* solution, its leading terms coinciding with the solution of the simplest model (2.10). The theory for nonlinear vorticity waves as the intermediate asymptotics was developed by Shrira (1989) for the most relevant geophysical situation, that of the shear flow localized near the surface. In particular, it was shown that the intrinsic dynamics of weakly nonlinear vorticity waves is governed by an essentially two-dimensional generalized Benjamin–Ono equation, rather than by the KP equation. Thus, vorticity waves represent wave-like perturbations in shear flows due to the inhomogeneous mean vorticity field supplied by the basic current. In the presence of a density stratification the vorticity and internal gravity waves interact and influence each other. This influence, being relatively weak far from the resonance, is greatly enhanced when the phase speed of the vorticity wave matches the celerity of one of the internal wave modes (Voronovich & Shrira 1996; Voronovich *et al.* 1997).

Thus, in the vicinity of the resonance, the evolution equation (2.7) is not applicable and, in the next section, we develop an asymptotic approach to derive new coupled

equations describing the resonance of finite-amplitude internal and vorticity waves. It is worthwhile mentioning that for the resonance to occur, the typical frequency of the vorticity waves should be of the same order as that of the internal waves N_0 (variables are dimensional)

$$\frac{U_0}{H_0} \sim N_0 = \max \left\{ \left(-g \frac{\rho_{0z}}{\rho_0} \right)^{1/2} \right\}. \quad (2.11)$$

The balance (2.11) being valid, the non-dimensional Brunt–Väisälä frequency N in (2.1b) is of order of unity in our scaling.

3. Asymptotic analysis

3.1. The core solution

Hereinafter we will confine ourselves to consideration of the shear flow localized in a thin subsurface layer of a width h , so that

$$h/H \sim \epsilon \ll 1, \quad (3.1)$$

and it is absent in the bulk of the fluid. Thus, the fluid is composed of two layers: the still core and a boundary layer with an effectively different scale of vertical motion. The natural way of treating the problem is to find a solution to (2.1)–(2.3) in the core subject to the no-flux condition at $z = -H$, then to introduce an inner-boundary-layer vertical variable

$$\zeta = z/\epsilon \quad (3.2)$$

and to find an inner solution subject to the ‘rigid–lid’ condition at $\zeta = 0$, and, finally, to match both solutions at $\zeta \rightarrow -\infty$, $z \rightarrow 0$.

To this end we introduce a set of ‘slow’ space–time coordinates

$$X = \mu(x - ct), \quad Y = \mu^2 y, \quad T = \mu^3 t, \quad (3.3)$$

where μ is a small parameter characterizing long spatial and slow temporal scales for the amplitude variations, while c is the long-wave speed limit of the internal wave mode subject to the resonance. To derive the evolution equations for wave amplitudes which would describe dispersive, diffractive, resonant, and nonlinear effects at the same order as the asymptotic expansion one has to assume a certain balance among the small parameters of the problem. The analysis of Voronovich & Shrira (1996) revealed that for the waves at resonance the appropriate balance is prescribed by the relation

$$\epsilon = \mu^4. \quad (3.4)$$

Thus, the solutions to the governing equations (2.1)–(2.3) should be looked for in the form of an expansion in powers of a single small parameter. The magnitudes of the motion in the bulk of the fluid and in the boundary layer are quite different and, thus, a particular amplitude scaling is required for these two layers.

In the bulk of the fluid we neglect the terms containing the mean flow in (2.1)–(2.3) and look for the normalized perturbation components

$$u = \epsilon \hat{u}, \quad v = \epsilon \mu \hat{v}, \quad w = \epsilon \mu \hat{w}, \quad p = \epsilon \hat{p}, \quad \rho = \epsilon \hat{\rho}, \quad (3.5)$$

where μ is related to ϵ through (3.4). Under the scaling (3.3)–(3.5) the primitive

equations (2.1) can be presented as a single equation for the vertical velocity \hat{w} :

$$\left(\hat{w}_{zz} + \frac{N^2}{c^2}\hat{w}\right)_{XX} + \epsilon^{1/2}\left[-\frac{2}{c}\hat{w}_{zzXT} - \hat{w}_{zzYY} + \hat{w}_{XXXX}\right] + O(\epsilon) = 0. \quad (3.6)$$

Solutions to this equation can be sought in the form of an asymptotic expansion,

$$\hat{w} = w_0 + \epsilon^{1/2}w_1 + O(\epsilon). \quad (3.7)$$

At the leading order the horizontal and vertical variables can be separated and the solution has a simple form

$$w_0 = -cA_X f, \quad (3.8)$$

where $A = A(X, Y, T)$ is the depth-independent amplitude of an internal wave, while $f = f(z)$ is the modal function satisfying the boundary-value problem

$$\left. \begin{aligned} f_{zz} + \frac{N^2}{c^2}f &= 0, \\ f(0) = f(-H) &= 0, \quad f_z(0) = 1. \end{aligned} \right\} \quad (3.9)$$

The last condition just specifies the normalization of the modal function. The boundary-value problem (3.9) (cf. (2.6)) is exactly the same as in studies of long internal waves propagating in a stratified Boussinesq fluid without shear (see e.g. Leblond & Mysak 1979). Thus, at the leading order the internal waves are not influenced by the shear flow and their speed and modal structure are completely prescribed by the density stratification. The effect of the shear flow on the internal waves occurs at the next order and results in a correction w_1 not satisfying the boundary condition (2.3) at $z = 0$, where the shear flow layer is located. Therefore, we are looking for the correction w_1 subject to the following boundary conditions:

$$w_1|_{z=0} = -cB_X, \quad w_1|_{z=-H} = 0, \quad (3.10)$$

where $B = B(X, Y, T)$ is the amplitude of the vorticity wave in the shear flow layer. Multiplying the $O(\epsilon^{1/2})$ terms of (3.6) by $f(z)$, integrating with respect to z and using (3.9), (3.10) we find an evolution equation for the internal mode amplitude A ,

$$\alpha(A_{TX} + \frac{1}{2}cA_{YY}) + \beta A_{XXXX} - c^2B_{XX} = 0, \quad (3.11)$$

where the coefficients α, β (cf. (2.8)) are given by

$$\alpha = 2c \int_{-H}^0 (f_z)^2 dz, \quad \beta = c^2 \int_{-H}^0 f^2 dz. \quad (3.12)$$

Equation (3.11) is the linearized version of (2.7) written for a particular internal mode having phase speed c . Thus, the internal wave is linear in this approximation, but being coupled with the vorticity wave, is, therefore, affected by its nonlinear behaviour. To close the system of amplitude equations we have to consider solutions in the boundary layer where the vorticity wave is located and to match them with the core solution taken at $z \rightarrow 0$ as follows:

$$u \rightarrow \epsilon c A + O(\epsilon^{3/2}), \quad (3.13a)$$

$$v \rightarrow \epsilon \mu c \left(\int A_Y dx \right) + O(\epsilon^{3/2} \mu), \quad (3.13b)$$

$$w \rightarrow -\epsilon^{3/2} \mu c B_X + O(\epsilon^2 \mu), \quad (3.13c)$$

$$p \rightarrow \epsilon c^2 A + O(\epsilon^{3/2}), \quad (3.13d)$$

$$\rho \rightarrow -\epsilon^{3/2} B \rho_{0z} \Big|_{z=0} + O(\epsilon^2). \quad (3.13e)$$

Note, that the perturbation field components in (3.13) are real non-dimensional variables, rather than the normalized ones having the hats in (3.5).

3.2. The boundary layer solution

Inside the boundary layer the scale of vertical motion is different from that in the bulk of the fluid. Therefore, both the mean flow velocity and the field variables are supposed to depend on the inner vertical variable ζ rather than on z , while the Brunt–Väisälä frequency is supposed to be constant $N = N(0)$ at this scale. The solution to the basic equations is subject to the ‘rigid-lid’ condition at $\zeta = 0$ and should be matched with (3.13) in the limit $\zeta \rightarrow -\infty$. The latter condition specifies the proper scaling of the field component amplitudes inside the boundary layer:

$$u = \epsilon^{1/2} \tilde{u}, \quad v = \epsilon \mu \tilde{v}, \quad w = \epsilon^{3/2} \mu \tilde{w}, \quad p = \epsilon \tilde{p}, \quad \rho = \epsilon^{3/2} \tilde{\rho}, \quad (3.14)$$

with the tilded variables being of order of unity.

Equation (3.14) indicates that fluid particles oscillate, basically in the streamwise direction, while vertical and spanwise motion is considerably weaker. This constitutes one of the main features of vorticity waves as it was first pointed out by Shrira (1989). Also, the resonance proved to lead to a significant (an order of magnitude) amplification of the horizontal motion near the surface compared to that at depth. Thus, in the presence of a subsurface shear current even a comparatively small-amplitude linear internal wave manifests itself as an intense, often nonlinear, wave-like perturbation of the flow (see Voronovich & Shrira 1996).

Equations (2.1) with (3.2)–(3.4), (3.14) taken into account can be reduced by virtue of a standard procedure to a single equation for the vertical velocity \tilde{w} :

$$(c - U) \tilde{w}_{\zeta\zeta X} + U_{\zeta\zeta} \tilde{w}_X + \epsilon^{1/2} \left[-\tilde{w}_{\zeta\zeta T} + (\tilde{u}\tilde{u}_X + \tilde{w}\tilde{u}_\zeta)_{\zeta X} \right] + O(\epsilon) = 0 \quad (3.15)$$

closed only in the leading-order (linear) approximation. It is worthwhile noting that the terms due to the density stratification do not appear at the leading orders in (3.15) and, thus, the restoring force in the upper layer is not gravity but the ‘vorticity’ force due to the mean shear. Mathematically this fact is a consequence of the smallness of the global Richardson number in the asymptotic limit under consideration

$$Ri \approx (N_0 h / V)^2 \ll 1,$$

in accordance with (2.11), (3.1).

We look for solutions to (3.15) in the form of power series in $\epsilon^{1/2}$:

$$\tilde{w} = w_1 + \epsilon^{1/2} w_2 + O(\epsilon), \quad (3.16a)$$

$$\tilde{u} = u_{-1} + \epsilon^{1/2} u_0 + O(\epsilon). \quad (3.16b)$$

The main-order solution for the vertical velocity is found from (3.15); other field components can be found from (2.1)–(2.3) and used for obtaining nonlinear terms in the next-order approximation.

Also, we suppose that the phase velocity of the vorticity wave c_v , which at the leading order equals that of the current at the surface, does not match exactly that of the internal wave c , but there is an $O(\epsilon^{1/2})$ mismatch Δc

$$c = c_v + \epsilon^{1/2} \Delta c. \quad (3.17)$$

On substituting (3.16) and (3.17) into (3.15) we find the leading-order terms of the velocity disturbance \tilde{w} and \tilde{u} in the form (Shrira 1989; Voronovich *et al.* 1997)

$$w_1 = -G_X(c_v - U), \quad u_{-1} = -GU_\zeta, \quad (3.18)$$

where G is the amplitude of the fundamental mode supported by the shear flow. Then, integrating the inhomogeneous equation for w_2 arising in the next-order approximation of (3.15) we find its general solutions in the form

$$w_2 = G_T - \Delta cG_X - GG_XU_\zeta + Q_X(c_v - U) \int^\zeta \frac{d\zeta'}{(c_v - U(\zeta'))^2}, \quad (3.19)$$

where Q is an amplitude of the second fundamental solution to the homogeneous problem following from (3.15). This fundamental solution is known in the theory of hydrodynamic stability as the Tollmien inviscid solution (Drazin & Reid 1979). Using (3.19), we also find the component u_0 of the horizontal velocity,

$$u_0 = \frac{1}{2}G^2U_{\zeta\zeta} + Q \left[U_\zeta \int^\zeta \frac{d\zeta'}{(c_v - U(\zeta'))^2} - \frac{1}{c_v - U} \right]. \quad (3.20)$$

Applying the matching conditions (3.13) we are able to express the unknown functions G and Q through the internal and vorticity wave amplitudes A and B :

$$G = B(X, Y, T), \quad Q = -c_v^2 A(X, Y, T). \quad (3.21)$$

On the other hand, applying the ‘rigid-lid’ condition at $\zeta = 0$ leads to an identity at the main order provided by $c_v = U|_{\zeta=0}$, but yields the second evolution equation for the wave amplitudes

$$B_T - \Delta cB_X - U'_0BB_X - \frac{U_0^2}{U'_0}A_X = 0, \quad (3.22)$$

where $U_0 = U|_{\zeta=0}$ and $U'_0 = U_\zeta|_{\zeta=0}$. In contrast to (3.11) the vorticity wave is governed by the nonlinear equation (3.22) and can exhibit quite strong nonlinear features as will be shown below.

The system of equations (3.11) and (3.22) governs the dynamics of the resonance between the vorticity wave and a long internal wave. This system reduces to the non-dimensional form (1.1) by means of a simple scaling transformation of dependent variables

$$A = -\frac{2U_0^2}{\alpha U'_0} a(x + \Delta t, y, t), \quad B = -\frac{2U_0^2}{\alpha^{1/2}(U'_0)^{3/2}} b(x + \Delta t, y, t),$$

and independent variables

$$\left. \begin{aligned} x &= \left(\frac{\alpha U_0^4}{\beta^2 U'_0} \right)^{1/4} X, & y &= \left(\frac{2U_0^3}{\beta U'_0} \right)^{1/2} Y, \\ t &= \left(\frac{U_0^{12}}{\alpha \beta^2 U_0^3} \right)^{1/4} T, & \Delta &= \left(\frac{\alpha U'_0}{U_0^4} \right)^{1/2} \Delta c. \end{aligned} \right\} \quad (3.23)$$

It should be mentioned that the streamwise velocity component u_0 is logarithmically divergent as $\zeta \rightarrow 0$ according to its asymptotic presentation

$$u_0 \rightarrow \frac{U''_0}{U_0^2} \left(\log |\zeta| + \frac{3}{2} \right) c_v^2 A + \frac{1}{2} U''_0 B^2, \quad (3.24)$$

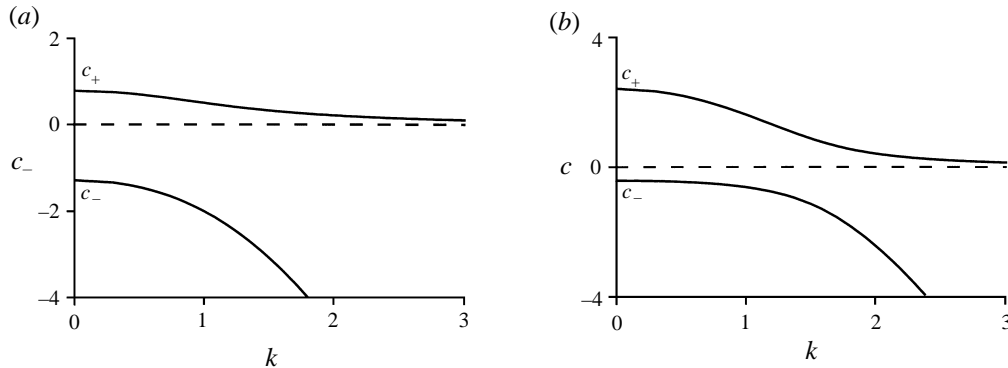


FIGURE 2. Splitting of dispersion curves in the vicinity of the resonance: (a) subcritical case ($\Delta = -1$); (b) supercritical case ($\Delta = 2$).

where $U_0'' = U_{0\zeta\zeta}|_{\zeta=0}$. This divergence indicates the existence of a *critical layer* at $\zeta = 0$, where $c_v = U(\zeta)$. Strictly speaking, in order to remove this divergence one has to modify the asymptotic technique and to introduce one more inner expansion, inside the critical layer. However, the analysis carried out by Voronovich *et al.* (1997) shows that the critical layer contribution is negligible at the order in which (3.22) was derived. This fact enables us to ignore the existence of the critical layer in our further consideration.

4. Plane solitary wave solutions

In the next two sections we confine ourselves to considering *plane* solitary waves within the framework of (1.1), i.e. we presume $A_{YY} \equiv 0$, leaving a brief discussion of some three-dimensional solutions to the system (1.1) to the Appendix. The basic system (1.1) is simplified to

$$\left. \begin{aligned} a_t + \Delta a_x + a_{xxx} - b_x &= 0, \\ b_t + 2bb_x - a_x &= 0. \end{aligned} \right\} \quad (4.1)$$

To get an idea of the dispersion properties of the system, first consider a solution to the linearized system (4.1) in the form of a pair of harmonic waves

$$a \sim \hat{a} \exp[ik(x - ct)], \quad b \sim \hat{b} \exp[ik(x - ct)],$$

where k is the wavenumber and $c = c(k)$ is the phase velocity. The dispersion relation for linear waves has two roots for each value of the wavenumber k :

$$c = c_{1,2} = \frac{1}{2} \left[\Delta - k^2 \pm (4 + (\Delta - k^2)^2)^{1/2} \right]. \quad (4.2)$$

Different roots correspond to the vorticity and internal waves modified by their interaction in the vicinity of the resonance. For linear waves the resonance results in a specific reconnection of the dispersion curves in the region of small k , which, in our case, does not lead to *linear* instability as (4.2) indicates. The parameter Δ specifies the phase velocity mismatch between two waves in the limit $k \rightarrow 0$. The dispersion curves for '*subcritical*' ($\Delta < 0$ or, equivalently, $c_v > c$) and '*supercritical*' ($\Delta > 0$ or $c_v < c$) resonances are plotted in figure 2. Equation (4.2) indicates (see also figure 2)

that for any Δ there exist two gaps, or forbidden zones, in the spectrum of the linear wave speeds:

$$c \in (c_-; 0), \quad c \in (c_+; \infty), \quad (4.3a)$$

where

$$c_{\pm} = \frac{1}{2} \left[\Delta \pm (4 + \Delta^2)^{1/2} \right]. \quad (4.3b)$$

Therefore, one may expect the existence of nonlinear solitary waves travelling with velocities that lie inside the forbidden zones. Otherwise, the resonance with infinitesimal linear waves would lead to a radiative damping of the solitary waves and, finally, to their decay. With this expectation in mind, we look for steady solitary solutions preserving their form and advancing with a constant speed

$$a = a_s(x - vt), \quad b = b_s(x - vt).$$

Our analysis is confined to the consideration of localized solutions having zero asymptotics at both infinities, i.e. $a_s, b_s \rightarrow 0$ as $x \rightarrow \pm\infty$. Then, integrating (4.1) we find the relation between a_s and b_s as

$$a_s = b_s^2 - vb_s \quad (4.4a)$$

and the ordinary differential equation to solve for the function b_s is

$$(v - 2b_s)^2 \left(\frac{db_s}{dx} \right)^2 = b_s^2 \left[(v - \Delta)b_s^2 + 2 \left(\frac{2}{3} - v(v - \Delta) \right) b_s + v(v(v - \Delta) - 1) \right]. \quad (4.4b)$$

Equation (4.4b) can be thought of as the energy integral of a particle moving in a potential well. From this point of view the coordinate x is an analogue of time and the amplitude b_s of coordinate, $(db_s/dx)^2$ represents the particle kinetic energy, the total energy of the particle is zero and the potential is given by the expression

$$\Pi(b_s) = - \frac{b_s^2}{(v - 2b_s)^2} \left[(v - \Delta)b_s^2 + 2 \left(\frac{2}{3} - v(v - \Delta) \right) b_s + v(v(v - \Delta) - 1) \right]. \quad (4.5)$$

The potential (4.5) specifies completely the phase plane of the particle (b_s, b'_s) , where $b'_s = db_s/dx$. Clearly, the origin of the coordinate frame $(0, 0)$ is always an equilibrium, but a homoclinic orbit, which originates at this point and represents a localized solitary wave solution, exists only if the point $(0, 0)$ is a saddle point, but not a centre. Thus, the potential should have a local maximum at $b_s = 0$ and, hence, its second derivative

$$\Pi''(b_s)|_{b_s=0} = -\frac{2}{v} (v^2 - \Delta v - 1) < 0 \quad (4.6)$$

should be negative. One can see that the condition (4.6) is fulfilled when the solitary wave speed v lies exactly inside the gaps in the spectrum of the linear waves (4.3).

Yet, for solitary wave solutions to exist, condition (4.6) is necessary but not sufficient. For a trajectory starting from the saddle point in the limit $t \rightarrow -\infty$ to return at $t \rightarrow +\infty$, a turning point where the potential changes its sign should exist. Only under this condition can a closed separatrix trajectory be formed. The turning point $b_s = b_{max}$ corresponds to the maximal amplitude of the solitary wave and is prescribed completely by the non-zero root of the equation $\Pi(b_s) = 0$ closest to the origin. In turn, the non-zero roots of $\Pi(b_s)$ coincide with those of the polynomial inside the brackets in (4.5). A simple analysis indicates that the roots exist if v lies inside the interval

$$v \in (v_-; v_+), \quad (4.7a)$$

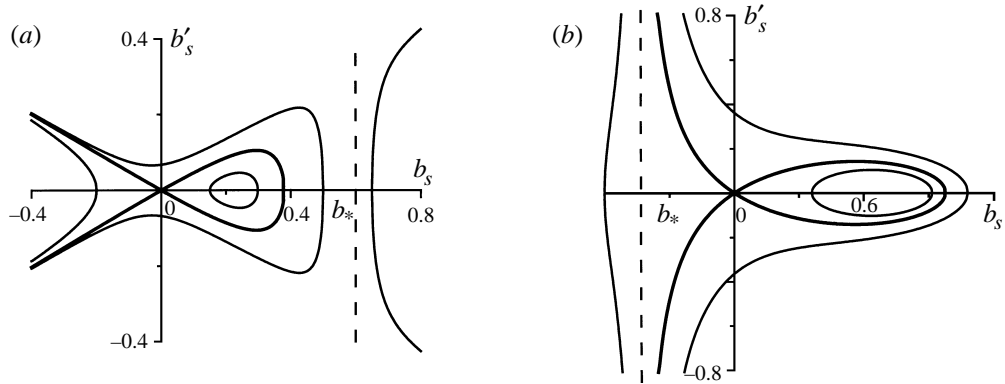


FIGURE 3. Phase plane for: (a) fast ($v = 1.2$) and (b) slow ($v = -0.75$) waves at $\Delta = 0$. The homoclinic orbits are shown by bold curves. The dashed lines display the values of b_s where the potential (4.5) is singular.

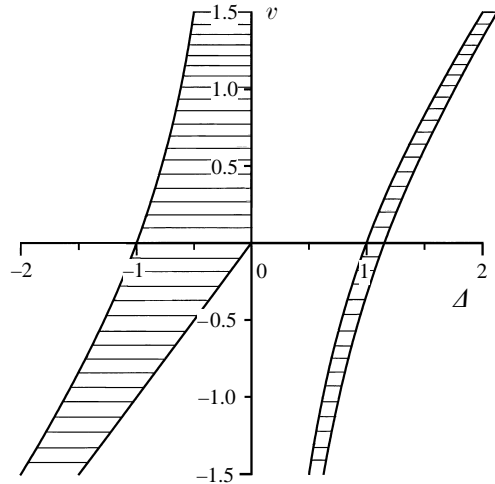


FIGURE 4. Domains of solitary wave existence on the (v, Δ) -plane.

where

$$v_{\pm} = \frac{1}{2} \left[\Delta \pm \left(\frac{16}{3} + \Delta^2 \right)^{1/2} \right]. \tag{4.7b}$$

Thus, we have two types of solitary waves in the model: the ‘fast’ ones, with velocity lying inside the interval $(c_+; v_+)$, and the ‘slow’ ones with negative velocity. Examples of the phase plane (b_s, b'_s) for the waves of both types are presented in figure 3, with the homoclinic orbits being marked by the bold curves. The regions of the (v, Δ) plane where the solitary wave solutions exist are plotted in figure 4.

The solitary wave solutions $b = b_s(x)$ can be found in closed form from (4.4b) by direct integration. First, consider the ‘fast’ solitary waves. They can be given by the following analytical expression:

$$\exp(-\kappa|x|) = \left[\frac{(b_-(b_+ - b_s))^{1/2} - (b_+(b_- - b_s))^{1/2}}{(b_-(b_+ - b_s))^{1/2} + (b_+(b_- - b_s))^{1/2}} \right] \left[\frac{(b_+ - b_s)^{1/2} + (b_- - b_s)^{1/2}}{(b_+ - b_s)^{1/2} - (b_- - b_s)^{1/2}} \right]^{2v}, \tag{4.8}$$

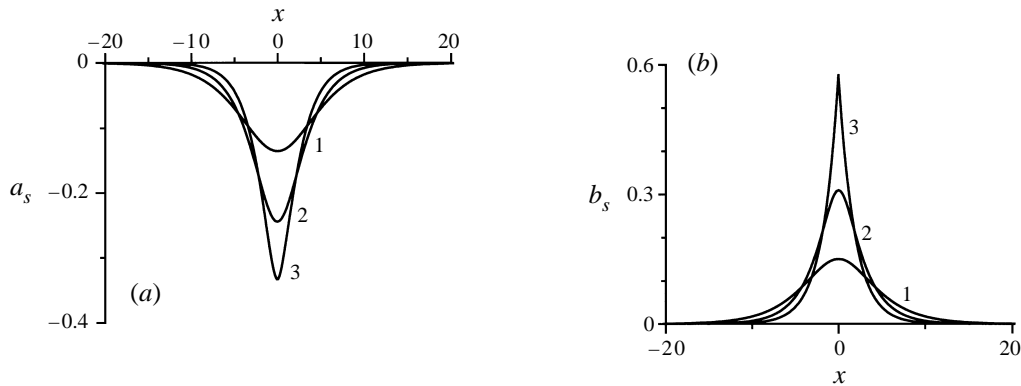


FIGURE 5. ‘Fast’ solitary wave profiles: (a) $a_s(x)$; (b) $b_s(x)$ for $\Delta = 0$. Curve 1, $v = 1.05$; curve 2, $v = 1.1$; curve 3, $v = 1.154$

where

$$\kappa = \left(\frac{v(v - \Delta) - 1}{v} \right)^{1/2}, \quad v = \left(\frac{v(v - \Delta) - 1}{v|v - \Delta|} \right)^{1/2},$$

and

$$b_{\pm} = \frac{3v(v - \Delta) - 2 \pm [4 - 3v(v - \Delta)]^{1/2}}{3(v - \Delta)}.$$

For $c_+ < v < v_+$ both non-zero roots of the nonlinear potential b_{\pm} prove to be positive and, moreover, direct calculations show that the inequality

$$b_- < \frac{1}{2}v \tag{4.9}$$

holds. Hence, the singularity in the potential $\Pi(b_s)$ occurring at $b_s = v/2$ does not prevent the solutions from existing, $b_s(x)$ is positive for all x while a_s according to (4.4a) is negative. Thus, fluid particles moving with the vorticity wave are accelerated while those moving with the internal wave are decelerated. The maximal amplitude of the solitary wave is specified by the smaller root of the potential: $b_{max} = b_-$. The profiles of b_s and a_s corresponding to the ‘fast’ solitary waves are plotted in figure 5(a, b). The dependence of the solitary wave velocity v and the characteristic width $\delta = \kappa^{-1}$ on the amplitude b_{max} is presented in figure 6(a, b) for several values of Δ . We would like to mention that similar dependencies were also found for the surface and interfacial solitary waves on a uniform shear flow (see figures 2 and 11 of Pullin & Grimshaw 1988).

The ‘fast’ solitary wave solutions of (4.4b) exhibit rather interesting limiting behaviour as their speed v tends to the edges of the existence interval. In the limit $v \rightarrow c_+$ the smaller root of the potential tends to coalesce with the saddle point, i.e. $b_- \rightarrow 0$, and the solitary wave (4.8) becomes asymptotically close to the small-amplitude KdV soliton, strongly elongated in the streamwise direction,

$$b_s \approx b_- \cosh^{-2} \left[\frac{1}{2} \kappa x \right], \tag{4.10a}$$

where b_- and κ are given by (4.8) and prove to be small in the limit $v \rightarrow c_+$. This limit corresponds to small-amplitude resonant vorticity and internal waves, the coupling of which and splitting of dispersion curves are due to the linear dispersive effects at the leading order. At the next order, taking account of nonlinearity leads to separate

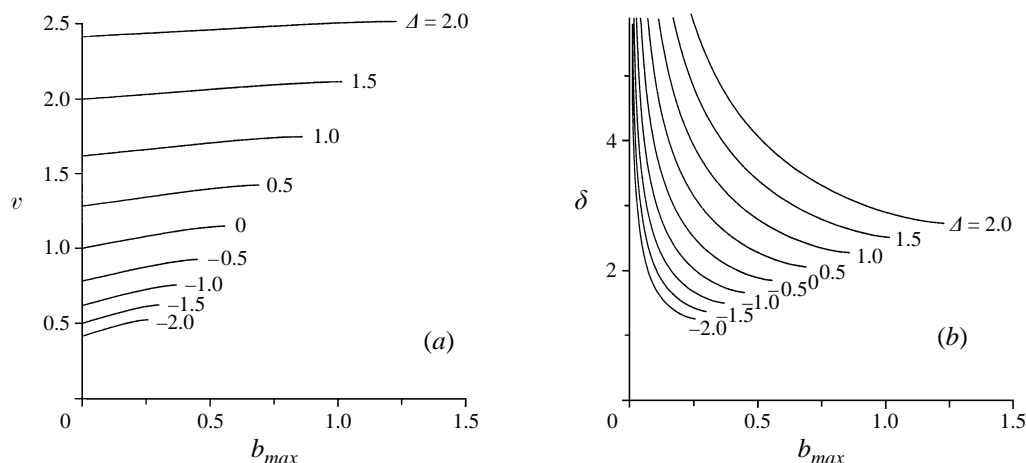


FIGURE 6. Solitary wave speed v (a) and width δ (b) vs. its amplitude b_{max} for 'fast' waves. Different curves correspond to different values of Δ as labelled.

effective KdV equations for both waves. In fact, soliton (4.10a) represents a solution of one of those asymptotic equations.

In the opposite limit $v \rightarrow v_+$ two non-zero roots of $\Pi(b_s)$ tend to coalesce and to form a double root at $b_{\pm} = b_* = v_+/2$, though the singularity which occurs exactly at this value makes the behaviour of the potential much more complex. If the wave speed v grows further, at $v > v_+$ the potential no longer has non-zero roots, and the separatrix trajectory originating from the saddle point in the phase plane (see figure 3a), therefore, goes to infinity and localized solitary solutions of (4.4b) no longer exist. Thus, the amplitude of the 'fast' solitary waves is bounded from above by the critical value $b_* = v_+/2$ (see figure 6a, b). The limiting solution can be found directly from (4.8) in a simple form,

$$b_s = b_* \exp(-\kappa_* |x|), \quad (4.10b)$$

where $\kappa_* = 1/(3v_+)^{1/2}$. This solution represents the solitary wave of the greatest height with a sharp corner at the crest. Its shape is plotted in figure 5(a, b) (curve 3) for the components $a = a_s$ and $b = b_s$, respectively. We note that the component a_s for the internal solitary wave remains smooth. The limiting solitary waves with a corner-type crest are often called *peaked* solitons and are a typical phenomenon in strongly nonlinear problems. Within the framework of weakly nonlinear theories the peaked solitons are very rare. Such solutions were first constructed analytically by Fornberg & Whitham (1978) in their studies of wave breaking by means of a model equation. As a matter of fact, their equation (29) possesses solitary wave solutions which differ from those of (4.4b) only in the numerical coefficients. An integrable evolution equation with solutions of the peaked-soliton type was derived recently by Camassa & Holm (1993).

Recall that in terms of the physical variables the amplitude a is proportional to the horizontal velocity u and the vertical displacement of the fluid particles. The vertical velocity w and the vertical acceleration w_t , being proportional to a_x and a_{xx} respectively, exhibit stronger singularities at the peaked solitary waves, which indicates that the derived system (4.1) is invalid for describing of the limiting solutions. If we note that in the dimensional variables the denominator in (4.4b) responsible for the singularity becomes proportional to $(u - V)$, where V is the dimensional celerity of

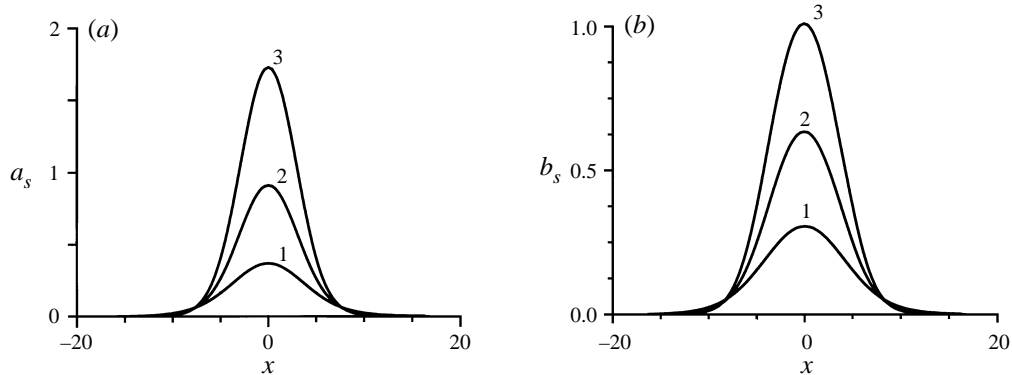


FIGURE 7. ‘Slow’ solitary wave profiles: (a) $a_s(x)$; (b) $b_s(x)$ for $\Delta = 0$ (curve 1, $v = -0.9$; curve 2, $v = -0.8$; curve 3, $v = -0.7$).

the solitons, then the specific physical mechanism responsible for the limitation of the wave amplitude of the fast solitary waves becomes clear: the horizontal velocity u of the fluid particle in the wave cannot exceed the wave phase velocity V . The fact that the presence of a thin shear layer severely diminishes the maximal wave amplitude is not surprising. A similar phenomenon occurs for water waves in the presence of even infinitesimally thin shear layer (Phillips 1977, §3.9).

If the solitary wave speed v lies inside the interval $(c_-; 0)$, i.e. the solitary wave is ‘slow’, the origin of the phase plane (b_s, b'_s) is again the saddle point (see figure 3b). The inequality $v_- < c_-$ indicates, according to (4.7), that the potential $\Pi(b_s)$ always has two non-zero roots b_{\pm} for $v \in (c_-, 0)$ which are given below (4.8). Yet, their location with respect to the point $b_s = 0$ depends on the relative values of v and Δ . If $v > \Delta$, which is possible only in the subcritical case, i.e. for negative Δ , both roots turn out to be negative, but the singularity always occurs between them and the origin, i.e.

$$b_{\pm} < \frac{1}{2}v.$$

A separatrix trajectory cannot cross this line† and, hence, the ‘slow’ solitary wave solutions do not exist for $\Delta < v < 0$.

If, now, the opposite inequality holds, i.e. $v < \Delta$, then the roots of $\Pi(b_s)$ have different signs so that $b_+ < 0$ and $b_- > 0$, while the singularity is located between the negative root b_+ and the origin. Therefore, a closed separatrix in the negative half-plane is prevented by the singularity of the potential, while this trajectory exists in the positive half-plane (see figure 3b). Thus, the ‘slow’ solitary waves exist for $v \in (c_-, \min(0, \Delta))$ (see figure 4). For these solitary wave solutions both a_s and b_s have positive polarity which corresponds to acceleration of fluid particles in the wave. Integrating (4.4b) we find the wave profile in the closed form

$$\exp(\kappa|x|) = \left[\frac{(b_-(b_s - b_+))^{1/2} - (b_+(b_s - b_-))^{1/2}}{(b_-(b_s - b_+))^{1/2} + (b_+(b_s - b_-))^{1/2}} \right] \exp \left[-4v \tan^{-1} \left(\frac{b_- - b_s}{b_s - b_+} \right)^{1/2} \right], \quad (4.11)$$

where κ , v , and b_{\pm} are the same as in (4.8), and we have used the inequality $b_+ < 0 < b_-$. Examples of the ‘slow’ solitary waves are depicted in figure 7(a, b)

† In this analysis, we do not consider solitary wave solutions having regions of infinite slope. Such singular solutions corresponding to separatrix trajectories crossing the singularities could be constructed for any value of $v \in (c_-, 0)$.

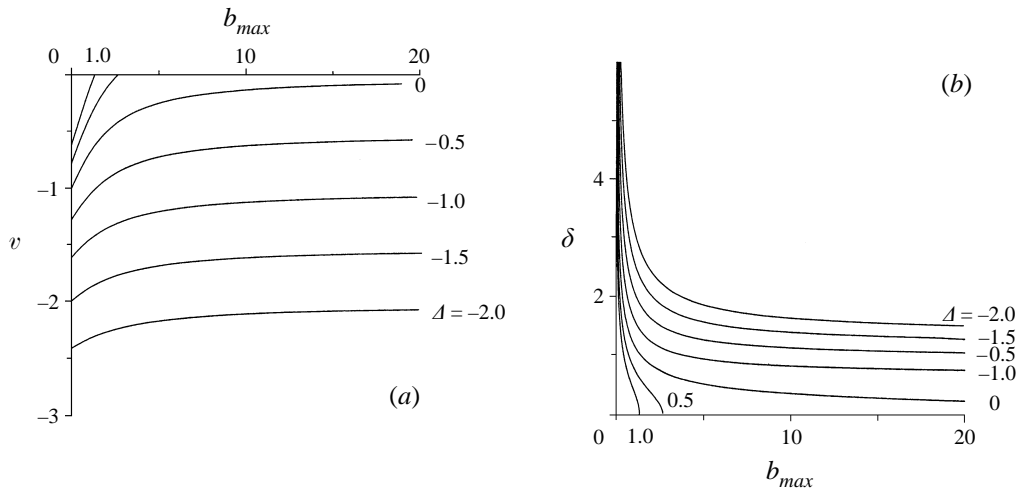


FIGURE 8. Solitary wave speed v (a) and width δ (b) vs. its amplitude b_{max} for 'slow' waves. Different curves correspond to different values of Δ .

for $\Delta = 0$, while the dependence of v and $\delta = \kappa^{-1}$ on the maximal amplitude $b_{max} = b_-$ is shown in figure 8(a, b) for several values of Δ . In the limit $v \rightarrow c_-$ the positive root of the potential again tends to zero and the amplitude of the solitary wave decreases. In this limit, the form of the solitary wave (4.11) tends asymptotically to that of the KdV soliton (4.10a) for the other effective KdV equation. On the other hand, for the subcritical case the amplitude of the 'slow' solitary waves is not bounded from above, i.e. $b_{max} \rightarrow +\infty$ for $v \rightarrow \Delta$. On the contrary, in the supercritical case the maximal amplitude of the wave is bounded by its limiting value $b_c = 4/(3\Delta)$ as $v \rightarrow 0$ (see figure 7a, b).

5. Stability and evolution

In this section we discuss the stability and evolution of the plane solitary wave solutions (4.8) and (4.11) as well as of general localized wave perturbations within our simplified system (4.1). First, we find two conserved quantities of this system which are the momentum and the energy of the nonlinear wave field:

$$P[a, b] = \frac{1}{2} \int_{-\infty}^{+\infty} (a^2 + b^2) dx, \quad (5.1a)$$

$$H[a, b] = \frac{1}{2} \int_{-\infty}^{+\infty} (a_x^2 - \Delta a^2 + 2ab - \frac{2}{3}b^3) dx. \quad (5.1b)$$

Equations (4.4a, b) for localized solitary wave solutions can be found from the *constrained* variational problem $\delta A = 0$, where A is the Lyapunov functional given by $A = H + vP$ and v serves as a Lagrange multiplier. In other words, the stationary solitary wave solutions can be regarded as extremal points of the energy functional H at a fixed value of P . In order to analyse the stability of these stationary solutions we introduce small (*infinitesimal*) perturbations, $\delta a = a - a_s$ and $\delta b = b - b_s$, which do not change the value of the momentum P . Therefore, these perturbations should satisfy the integral constraint

$$F[\delta a, \delta b] = \int_{-\infty}^{+\infty} (a_s \delta a + b_s \delta b) dx = 0. \quad (5.2)$$

Then, we evaluate from (5.1) the second variation of the Lyapunov functional $\delta^2\mathcal{A}$ which has the form

$$\delta^2\mathcal{A} = \frac{1}{2} \int_{-\infty}^{+\infty} [\delta a L \delta a + (v - 2b_s)(\delta\tilde{b})^2] dx, \tag{5.3}$$

where $\delta\tilde{b} = \delta b + (v - 2b_s)^{-1}\delta a$ and the linearized operator L is given by

$$L = -\frac{\partial^2}{\partial x^2} + (v - \Delta) - \frac{1}{v - 2b_s}. \tag{5.4}$$

Stability of soliton solutions in the energetic sense means that the second variation of the Lyapunov functional $\delta^2\mathcal{A}$ (5.3) is *sign-definite* (say, positive-definite) under the constraint (5.2) (see e.g. Grillakis, Shatah & Strauss 1987). In the opposite case, when the second variation is sign-indefinite, the soliton solutions are unstable and the small perturbations δa and δb grow exponentially in time.

Let us consider the ‘fast’ solitary wave solutions (4.8) for which $c_+ < v \leq v_+$ and $(v - 2b_s) \geq 0$ for any x . Then, the second term in (5.3) is non-negative. On the other hand, the operator L is a standard Sturm–Liouville operator with the known neutral mode, $L\chi_0 = 0$, where $\chi_0 = a_{sx}$. The function a_{sx} is odd with only one node and, therefore, according to the oscillation theorem, the operator L has a unique localized ground-state mode χ_- with a negative eigenvalue $\mu_- < 0$, $L\chi_- = \mu_- \chi_-$. The rest of the spectrum, $L\chi_+ = \mu_+ \chi_+$, contains continuum-spectrum eigenfunctions $\chi_+(p)$ corresponding to the eigenvalues $\mu_+(p)$ bounded from below, as $\mu_+(p) \geq \mu_+(0) = v - \Delta - v^{-1} > 0$. Other localized modes are also possible in the gap for μ_+ between zero and $\mu_+(0)$. For simplicity we suppose that these additional modes are absent although our analysis is valid for the general case as well. Also, we assume that the functions χ_- , χ_0 , and $\chi_+(p)$ form a complete orthogonal and orthonormal basis.

To find the criterion for soliton stability we apply the well-known method of Lagrange multipliers which was originally proposed by Vakhitov & Kolokolov (1973) in their studies of soliton stability within the framework of the nonlinear Schrödinger equation. Following their analysis, we consider the minimizing problem for the functional $\delta^2\mathcal{A}$ (5.3) constrained by (5.2).

This problem reduces to finding the minimal eigenvalue λ for the following system:

$$L\delta a_m = \lambda\delta a_m + v \left(\frac{b_s}{v - 2b_s} - a_s \right), \tag{5.5a}$$

$$\delta\tilde{b}_m = -\frac{vb_s}{v - 2b_s}, \tag{5.5b}$$

where v is the Lagrange multiplier for the minimizing problem. If the minimal eigenvalue is negative, then the second variation $\delta^2\mathcal{A}$ has a negative subspace, i.e. it is sign-indefinite, which indicates soliton instability. Using the properties of the linearized operator L we express solutions to (5.5a) through the orthonormal set of the eigenfunctions, χ_- , χ_0 , and $\chi_+(p)$:

$$\left. \begin{aligned} a_s - \frac{b_s}{v - 2b_s} &= r_- \chi_- + \int_0^\infty r(p) \chi_+(p) dp, \\ \delta a_m &= v \left[\frac{r_-}{\lambda - \mu_-} \chi_- + \int_0^\infty \frac{r(p)}{\lambda - \mu(p)} \chi_+(p) dp \right], \end{aligned} \right\}, \tag{5.6}$$

where r_- and $r(p)$ are spectral coefficients of the expansion. It can be easily shown

that the function χ_0 does not enter this expansion because of the different parity of the functions a_s and a_{sx} . Using this representation we rewrite the integral constraint (5.2) as a function of λ :

$$F(\lambda) = v \left[\frac{r_-^2}{\lambda - \mu_-} + \int_0^\infty \frac{r^2(p)}{\lambda - \mu_+(p)} dp - \int_{-\infty}^{+\infty} \frac{b_s^2}{v - 2b_s} dx \right]. \quad (5.7)$$

The minimal eigenvalue λ is a root of the equation $F(\lambda) = 0$. It is obvious that λ is bounded from below by μ_- , which can be reached only for $v = 0$ and $r_- = 0$. We assume here that $v > 0$ (without loss of generality) and also $r_- \neq 0$; the latter condition is equivalent to the following integral restriction:

$$\int_{-\infty}^{+\infty} \chi_- \frac{\partial a_s}{\partial v} dx \neq 0. \quad (5.8)$$

Under this assumption we can see from (5.7) that λ is located between $\mu_- < 0$ for which $F(\lambda \rightarrow \mu_- + 0) \rightarrow +\infty$ and $\mu_+(0) > 0$ for which $F(\lambda \rightarrow \mu_+(0) - 0) \rightarrow -\infty$. Therefore, a root $F(\lambda) = 0$ exists inside this interval. Also, this root is unique because the function $F(\lambda)$ is continuous and decreasing in a monotonic manner (see (5.7)). Hence, the sign of λ is controlled by the sign of $F(0)$ so that their signs coincide. Using (5.2), (5.5b) and the properties of the operator L we can evaluate this quantity in the explicit form

$$F(0) = v \frac{dP_s}{dv}, \quad (5.9)$$

where $P_s = P_s(v)$ is defined by (5.1a) for $a = a_s$ and $b = b_s$. Thus, we prove that the minimal eigenvalue is positive for

$$\frac{dP_s}{dv} > 0, \quad (5.10)$$

and, therefore, the ‘fast’ solitary wave solutions (4.8) are stable under the condition (5.10). In the opposite case, when $dP_s/dv < 0$, the minimal eigenvalue is negative and the solitary wave solutions are linearly unstable. In the critical case, $dP_s/dv = 0$, it follows from (5.5b) that the perturbations minimizing $\delta^2 \mathcal{A}$ lead to renormalization of the solitary wave parameter v according to

$$\delta a_m = r_0 a_{sx} + v \frac{\partial a_s}{\partial v}, \quad \delta b_m = r_0 b_{sx} + v \frac{\partial b_s}{\partial v}, \quad (5.11)$$

where r_0 and v are arbitrary parameters. In this case, the infinitesimal perturbations δa and δb grow in time in a power-like manner: $\delta a, \delta b \sim O(t^2)$ as $t \rightarrow \infty$ (see Pelinovsky & Grimshaw 1997 and references therein).

We display the dependence $P_s(v)$ calculated for the solitary wave solutions of the first branch for several values of Δ in figure 9(a–c). As a matter of fact, for *any* Δ there is a very narrow region $v_c < v \leq v_+$ near the limiting peaked soliton (4.10b) where the solitary waves (4.8) are unstable. This instability region is difficult to distinguish for $\Delta \leq 0$ (see figure 9a) but for large positive Δ this region is clearly visible in figures 9(b, c).

Outside this region the soliton solutions remain stable including the asymptotic limit $v \rightarrow c_+$ in accordance with the well-known results on stability of the KdV solitons (4.10a) (see e.g. Pelinovsky & Grimshaw 1997).

Consider now the ‘slow’ solitary wave solutions (4.11) for which $c_- < v < \min(0, \Delta)$ and $(v - 2b_s) < 0$. In this case, the properties of the operator L are the same as above but the second term in (5.3) is always negative. Therefore, there exist not

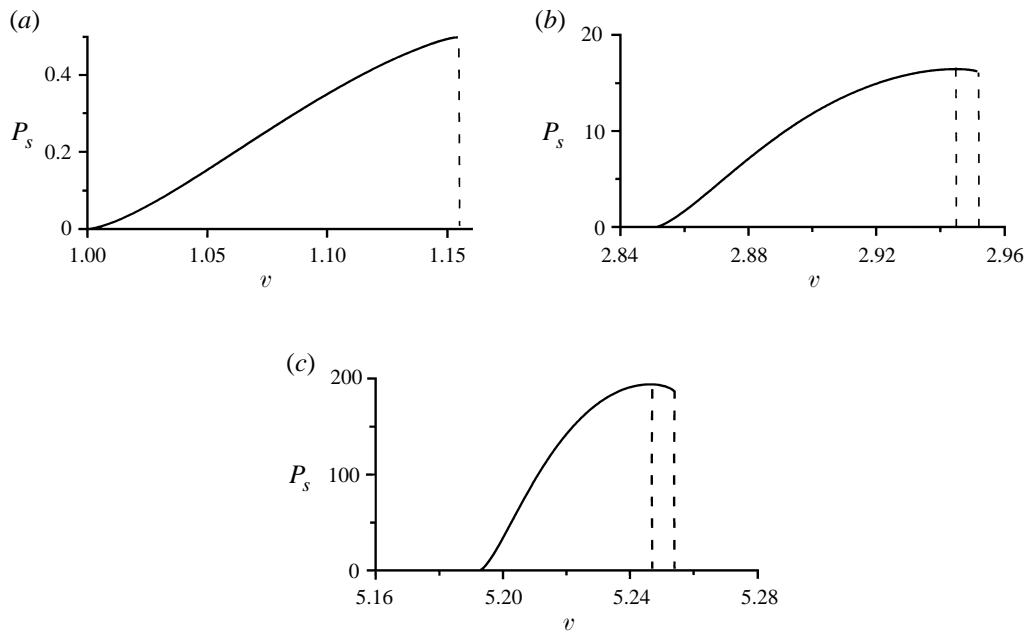


FIGURE 9. Functions $P_s(v)$ for different values of the phase mismatch: (a) $\Delta = 0$; (b) $\Delta = 2.5$; (c) $\Delta = 5$.

only the localized mode χ_- with a negative value of the second variation $\delta^2\mathcal{A}$ but also an infinite-dimensional subspace of soliton variations. In this case, the analysis described above and the general stability theorem are not applicable and the soliton solutions might be unstable even if the criterion (5.10) is satisfied (see Pego, Smereka & Weinstein 1995 for an example of such instability). We have checked that the slope $P_s(v)$ for the soliton solutions (4.11) is always positive but this does not exclude the possibility of oscillatory type instabilities. The latter problem needs a special mathematical study and lies beyond the scope of the present paper. It seems more important to show that the stability properties of the soliton solutions can lead to wave breaking within the framework of system (4.1) and, therefore, limit applicability of our weakly nonlinear asymptotic approach developed in §3.

In the rest of this section we discuss the evolutionary properties of localized initial perturbations and the possible formation of solitary waves or singularities, of the ‘vertical slope’ type, within the framework of the system (4.1). First, we mention that the development of instability for solitary waves of different classes was recently described in the review article by Pelinovsky & Grimshaw (1997). Their approach is valid in the vicinity of the instability threshold, which is realized in our context for special values of the soliton amplitude, where the slope of the function $P_s(v)$ is small. Near this instability threshold, the *weak* instability of the solitary wave solutions leads to their *adiabatic* evolution, i.e. the solitary wave as it evolves remains self-similar to the stationary wave profile with the parameter v varying in time. Indeed, we have seen from (5.11) that the perturbation weakly growing at the soliton background changes the soliton parameters. As a result of this adiabatic evolution, the amplitude of the solitary waves grows and reaches the critical value b_* corresponding to the limiting peaked soliton (4.10b). As the peaked soliton is also unstable, the solitary wave evolution cannot be stabilized by a sharp corner wave profile. Hence we expect

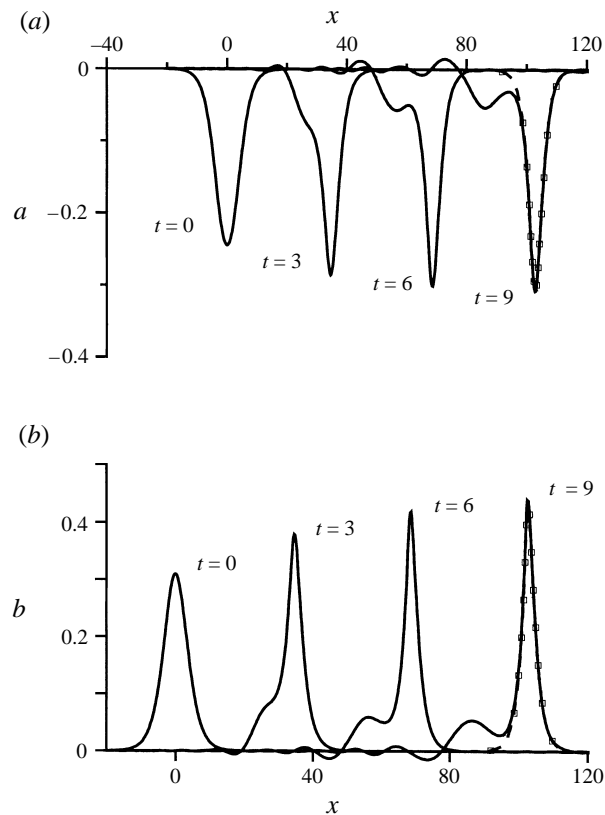


FIGURE 10. Nonlinear evolution of an initial pulse of subcritical amplitude. The profile of the relevant soliton solution of family (4.8) is shown for comparison by the dashed line.

that further evolution would lead to the formation of vertical slopes for the solitary wave profile which indicates wave breaking.

On the other hand, the solitary wave solutions are stable in the parameter domain $c_+ < v < v_c$ (see figure 9a–c) and, therefore, these solitons might attract certain localized perturbations playing the role of elementary particles in the nonlinear wave-field dynamics. The emergence of solitary waves as the intermediate asymptotics of ‘small-amplitude’ (subcritical) localized perturbations could be traced numerically within the framework of the system (4.1). For ‘large-amplitude’ (supercritical) initial perturbations the situation seems to be more complicated: the direct formation of the solitons is unlikely and, instead, we can expect wave breaking developing out of the large-amplitude perturbations. Possibly, after wave breaking, solitary waves could still be formed but this scenario cannot be adequately described by our asymptotic system (4.1).

To check these qualitative predictions of the stability analysis some preliminary numerical simulations of the system (4.1) were produced for $\Delta = 0$ under the periodic boundary conditions. The initial perturbation was taken in the form of (4.8) with the amplitude $b_{max} = b_-$ and the parameter κ given below (4.8). The time evolution for the case $v = 1.1$ and $b_{max} \approx 0.3$ is displayed in figure 10(a, b) and, as we have expected, results in the formation of a solitary wave with the amplitude $b_{max} \approx 0.45$.

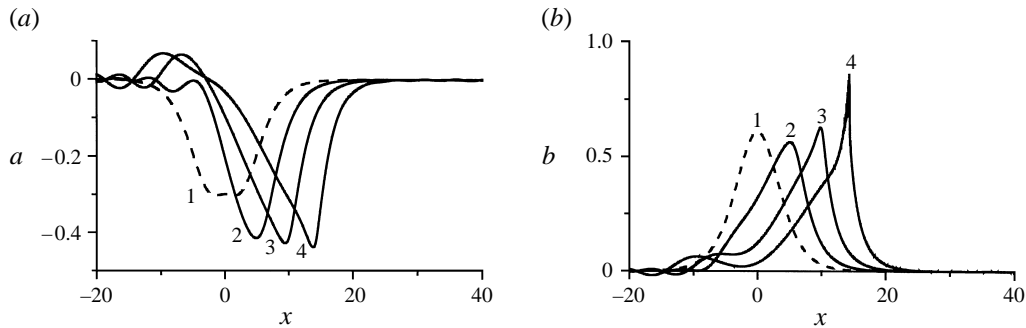


FIGURE 11. Nonlinear evolution of an initial pulse of supercritical amplitude.

For this case, the critical amplitude is $b_* \approx 0.57$ so that the solitary wave formed in this process is strongly nonlinear.

These results indicate that solitary waves even of near-critical amplitudes are the stable elements of the nonlinear wave-field dynamics.

Another simulation shown in figure 11(*a, b*) was made for the same initial conditions (curve 1) but with the amplitude b_{max} exceeding the critical value given by the exact analytical formula. We took $b_{max} \approx 0.6$, which means that the amplitude is slightly supercritical. It can be clearly seen from figure 11(*a*) that the evolution of the localized perturbation leads to the formation of vertical slopes after which the numerical solution breaks down at time $t \approx 8.5$ (curve 4). We conclude that the evolution of localized perturbations with supercritical amplitudes leads to wave breaking in accordance to our expectations. The question of whether a smaller-amplitude solitary wave emerges after the breaking event cannot be resolved within the system (4.1).

6. Discussion

The main conclusion of the paper is that the presence of the subsurface shear current may change drastically the nonlinear dynamics of internal waves in shallow water. Provided the resonance condition is met the interaction with the vorticity mode provides the small-amplitude internal mode with nonlinearity. To describe the coupled dynamics of the internal gravity and vorticity modes at resonance we have derived a new system of evolution equations. This model inherits some properties of the coupled KdV (KP) equations, on the one hand, and the integro-differential Whitham equation, on the other hand. Its analysis of preliminary character, which we carried out, was focused on the properties of the basic solutions – the solitary waves. The two classes of solitary waves explicitly found exhibit quite different properties: for the ‘fast’ solitons there is a critical amplitude they cannot exceed and the corresponding limiting shape is characterized by a sharp corner at the crest, while the ‘slow’ ones, being always smooth, have no limitations on the wave amplitude. The ‘fast’ solitons proved to be stable unless their amplitudes are very close to the limiting one. Numerical simulation shows that such solitons can be formed from localized perturbations of subcritical amplitudes. The localized pulses of larger, supercritical, amplitude develop vertical slopes in finite time which indicates wave breaking.

These two scenarios of localized pulse dynamics resemble those of wave evolution within the Whitham equation (see figures 13 and 14 of Fornberg & Whitham 1978). The formation of infinite slopes in finite time described by our evolution model implies that the small-but-finite-amplitude asymptotic approach loses its validity in

the vicinity of the wave breaking. In this case, Amick & Turner (1986) showed that either a vertical tangent and overhanging regions should develop in bubble-capped wave structures (see Pullin & Grimshaw 1988) or the fluid system allows the formation of an internal bore. The description of such strongly nonlinear vortex structures goes far beyond the aims of the present paper.

Before turning to the discussion of the possible implications of the above results it seems helpful to give a better idea of the typical scales involved. Let us consider the relevant parameters specified by the resonance condition of matching of the current velocity at the surface, U_0 , and the phase velocity of the long internal waves c , estimating the latter roughly as $N_0H/\pi n$, where N_0 is the depth-averaged Brunt–Väisälä frequency, H is the characteristic fluid depth and n is the vertical mode number. It is easy to see that for the typical velocity and the Brunt–Väisälä frequency N_0 being, say, $\approx 5 \times 10^{-2} \text{ m s}^{-1}$ and $1\text{--}3 \times 10^{-3} \text{ s}^{-1}$, respectively, the resonance condition selects depths of about 50–150 m for the main mode in the case of collinear propagation. Obviously, the higher modes become resonant at n -times larger depths, while the oblique orientation of the internal wave and current requires smaller depths. Thus, the typical resonant depths fall into the range of a few dozens to 100–200 m. Note, that to be strong the resonance interaction does not require large-amplitude internal waves, which allows the waves of higher modes, usually barely seen in the observations, to interact intensively with the current.

The accepted scaling (3.4) prescribes the typical internal wavelengths and amplitudes involved. We would like to emphasize that the key small parameter of our asymptotic expansion is μ^2 and not ϵ or μ ; the latter should not be small in itself. Say that under the physical conditions specified above for $H \approx 10^2 \text{ m}$ and the small parameter $\mu^2 = \epsilon^{1/2} \approx 10^{-1}$ one gets an estimate of the typical wavelength being $\approx 2 \text{ km}$ and of the current vertical scale of order of a few metres. The corresponding wave amplitudes (in terms of the isopycne displacements) are of order of one metre. These values are very typical of internal waves observed on the shelf and of wind-generated drift currents.

The underlying scaling (3.4) of the model assumes a balance of the nonlinearity of the vorticity mode and dispersion due to non-hydrostatic effects for the internal mode in the resonance. The diffraction effects are presumed to be of the same order. What happens if the scales involved are different and the balance doesn't hold? Are there other relevant balances? There are two main distinct situations. The picture is clear when the amplitude of the internal wave is much smaller, i.e. is $o(\epsilon)$, while the amplitude of the vorticity mode remains of the same order. Then the term a_x providing the coupling with the internal mode drops out of the equation for the vorticity mode (the second equation in the set (1.1)). The resulting decoupled Riemann wave equation predicts wave steepening and the 'gradient catastrophe' after a while, unless wave dispersion is taken into account. As soon as dispersion is accounted for we arrive at the model describing the vorticity wave evolution with weak nonlinearity and *its own* dispersion being in balance. The model was derived and studied in Voronovich *et al.* (1997). Within the framework of this model small-amplitude resonant internal waves are either emitted or absorbed by the nonlinear vorticity mode. The situation when the amplitude of the internal wave mode greatly exceeds ϵ is much more difficult to analyse. Say, if we start with the standard KdV-type scaling for the internal mode, i.e. presume weak linear dispersion balanced by the mode's own quadratic nonlinearity, this implies velocities in the water interior due to the wave to be $u = O(\mu^2) = O(\epsilon^{1/2})$ and $w = O(\mu^3)$. In virtue of continuity of the vertical displacement the scaling of w holds in the surface layer as well, where the continuity equation indicates that

the balance $w = O(\mu\epsilon u)$ is relevant, which means $u = O(1)$, i.e. we arrive at *strongly nonlinear* motion. Thus, one may expect internal waves obeying the KdV-type balance to induce violent motions in the surface layer. These though being hardly tractable analytically will on the other hand produce strong easily visible surface signatures.

Some immediate implications of the results of the paper may be summarized as follows:

(i) The resonance mechanism investigated links the processes in the water interior to near-surface motions. Even small variations in the surface current velocity noticeably modulate the intensity of short gravity and gravity–capillary wind waves and thus create signatures easily detectable by microwave radars and by the unaided eye (e.g. Robinson 1985). The asymptotic solutions derived enable one to describe quantitatively the amplification of the internal wave manifestations beyond the range of validity of linear theory.

(ii) The theory predicts the appearance of new types of internal solitary waves in the shelf zone, quite distinct from the KdV and mKdV solitons. Thus, it may be helpful in interpreting field data that do not fit the classical models.

The analysis of the derived equations carried out in the paper is in no way exhaustive. The consideration was confined to the plane solitary wave solutions and their straightforward oblique generalizations (see the Appendix). The oblique solitons obtained enable one to construct the resonant soliton triads to model the Mach stem phenomenon for the waves under consideration similarly to Miles (1977). The picture of such interactions is expected to be very rich, because of the non-trivial dependence of the solitary wave width and velocity on the amplitude and direction, and, therefore, in our opinion, it merits a special investigation.

In our study of the phenomenon of the resonance, for the sake of simplicity and clarity of presentation, we have neglected one of the major factors usually influencing internal wave dynamics in the shelf zone – the depth variations. Taking into account the smooth non-uniformity of the water depth and, if necessary, of the surface current, although straightforward, entails more cumbersome formulae in the derivation process. The resulting set of equations differs from the system (1.1) only in the slow time dependence of the coefficients. Although the complete analysis of such a system is hardly possible at present, an important class of problems concerned with the adiabatic evolution of solitary waves can be successfully treated by the available perturbation techniques utilizing the closed form solutions obtained. This, however, will be a subject of a separate investigation.

It should be mentioned that carrying out a quite similar analysis for the case of the interaction between an internal gravity mode and bottom boundary current leads to the same system (4.1) for the plane waves but does not allow one to obtain its weakly two-dimensional version (1.1). This is due to the qualitative difference in the boundary conditions at the bottom and the free surface for the viscous modes essential for regularizing the non-planar solutions (see Voronovich *et al.* 1997 for details). The intensification of the near-bottom motions by small-amplitude internal waves seems to be of importance in the context of sediment transport.

A number of less straightforward implications of the paper may be found by applying the main ‘physical’ idea of the work – consideration of the nonlinear wave–critical layer interaction as a wave–wave one – to some other situations where the shear flow is strongly localized and stable, the concept of vorticity waves is applicable and a weakly dispersive mode of a different nature exists and admits direct resonance. We mention just a few such examples, seemingly the most interesting in the context of geophysical fluid dynamics: wave–shear current interaction in the deep ocean

for the situations where the subsurface current is in resonance with internal waves governed by a Benjamin–Ono-type equation; in very shallow basins the dispersion effects due to rotation may prevail and such a situation may result in a similar direct resonant coupling of the vorticity mode and an internal gravity one described by the Ostrovsky equation (see Ostrovsky 1978; Ostrovsky & Stepanyants 1989) with or without KdV-type dispersion. At larger scales similar resonance may occur for continental shelf waves and vorticity waves in the boundary layer currents of the type described recently in Shrira & Voronovich (1996). Despite some mathematical similarity provided by the same underlying idea the above problems are physically essentially distinct and each requires a special study of its own.

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Appendix. Three-dimensional wave motion

The full three-dimensional nonlinear system (1.1) is much more complex to investigate than its reduced two-dimensional version (4.1): its detailed study lies beyond the scopes of the present work. While the reduced system has at least four conserving quantities: $P[u, w]$, $H[u, w]$ given by (5.1) plus the integrals of the wave masses,

$$\mathcal{A} = \int_{-\infty}^{+\infty} a \, dx, \quad \mathcal{B} = \int_{-\infty}^{+\infty} b \, dx, \quad (\text{A } 1)$$

we have not found any explicit integrals for the full system yet. Still some useful information on the nature and prospective regimes of wave dynamics can be extracted from the system (1.1) by comparatively simple means. Let us again consider a perturbation consisting of a pair of oblique harmonic waves of very small amplitude:

$$a \sim \hat{a} \exp[ik(x - ct) + ipy], \quad b \sim \hat{b} \exp[ik(x - ct) + ipy].$$

Substitution into (1.1) yields the linear dispersion relation

$$c = c_{1,2} = \frac{1}{2} \left[\Delta - k^2 + p^2 \pm (4 + (\Delta - k^2 + p^2)^2)^{1/2} \right]. \quad (\text{A } 2)$$

Straightforward calculations immediately indicate that unlike (4.2) solutions of (A 2) can have an arbitrary value and, thus, there are no gaps in the spectrum of the linear waves in the system. Thus, a fully localized three-dimensional solution of (1.1) moving with an arbitrary speed would always be in resonance with some small-amplitude wave, the result being the emission of resonant linear waves and, therefore, radiative damping of the three-dimensional wave (see Voronovich *et al.* 1997 for a thorough discussion). So, fully localized stationary solutions to (1.1) are very unlikely to exist.

It is worthwhile noting that the likely absence of three-dimensional solitary waves may be explained based on the specific dispersive properties of *internal* waves which lead to the squares of the stream- and spanwise components of the wavevector entering (A 2) with different signs. If these properties were different and be the signs

in front of k^2 and p^2 the same, the gaps in the spectrum of linear oblique waves would exist as well, resulting, probably, in solitary waves localized in all directions. In contrast, *vorticity* waves occurring in a shear flow of uniform density can exist in the form of three-dimensional solitary waves or ‘spikes’, an example being constructed numerically in Voronovich *et al.* (1997).

Still, the system (1.1), except for plane solitary waves advancing streamwise, admits similar solutions propagating in oblique directions. The latter can be easily found by looking for solutions of the form

$$a = a_s(x + qy - vt), \quad b = b_s(x + qy - vt), \quad (\text{A } 3)$$

where $q = \tan \theta$ and θ is the angle between the directions of the mean flow velocity and that of the wave propagation. Substitution of (A 3) into (1.1) and successive integration gives us exactly (4.4) with the renormalized mismatch parameter

$$\Delta \rightarrow \Delta_q = \Delta + q^2. \quad (\text{A } 4)$$

An analysis of the nonlinear potential, which in this case has an additional parameter q , indicates that again there exist two families of steady solitary waves. The ‘fast’ ones propagate with the constant velocity lying inside the interval $(c^+; v^+)$, while the speed of the ‘slow’ ones lies in the interval $(c^-; \max\{0, \Delta_q\})$ where now

$$c^\pm = \frac{1}{2} \left[\Delta + q^2 \pm \left(4 + (\Delta + q^2)^2 \right)^{1/2} \right] \quad (\text{A } 5a)$$

and

$$v^+ = \frac{1}{2} \left[\Delta + q^2 + \left(\frac{16}{3} + (\Delta + q^2)^2 \right)^{1/2} \right]. \quad (\text{A } 5b)$$

With Δ fixed, one can easily see from (A 5) that as the angle of wave propagation θ grows from zero to $\pm\pi/2$ the values of c^\pm, v^+ also grow. The limiting behaviour at very large values of $q = \tan \theta$ is given by the expressions

$$c^+ \rightarrow q^2 + \Delta + \frac{1}{q^2}, \quad v^+ \rightarrow q^2 + \Delta + \frac{4}{3q^2}, \quad c^- \rightarrow -\frac{1}{q^2} \quad \text{at } |q| \rightarrow \infty. \quad (\text{A } 6)$$

Therefore, oblique solitons, in general, prove to advance with greater speeds than the streamwise ones, while the gaps in the spectrum, where they can exist, become more and more narrow (see figure 6). It is interesting to note that the limiting amplitude of the ‘fast’ solitons which is still equal to $v^+/2$ also grows, so the oblique solitary waves may have larger amplitudes than those moving exactly streamwise.

As the scaling (3.3) was based on the assumption of the spanwise motion scale being much larger than the streamwise one, it is understood that the asymptotic procedure leading to (1.1) is valid only at moderate values of q and therefore at angles not too close to $\pi/2$.

The simple solutions obtained above by renormalization of parameters of the streamwise plane solitary waves provide the building blocks for constructing some truly three-dimensional solutions of the resonant triad type first considered for the KP solitons. The picture of such interactions is expected to be much richer than within the framework of the KP equation because of the more complicated dependence of the solitary wave width and velocity on the amplitude and direction of wave propagation.

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